

# Guaranteed control design under $L_p$ -compact constraints on the disturbance

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**Abstract:** The paper deals with the problem of optimization of a guaranteed (worst case) result for a control system described by an ordinary differential equation. The disturbances as functions of time are subject to functional constraints belonging to a given family of constraints. The latter family is known to the side that forms the control actions. The controlling side uses positional full-memory strategies and does not observe the disturbance. When the constraints family consists of  $L_p$ -compact sets the optimal guaranteed result is non-improvable in the sense that it coincides with that obtained in the class of quasi-strategies – nonanticipatory transformations of disturbances into controls.

In this paper for the effectiveness of implemented control algorithm an additional condition on the system and appropriate ways of constructing an optimal strategy are specified.

*Keywords:* Optimal guaranteed result, full-memory control strategies, functionally constrained disturbances, quasi-strategies.

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## 1. INTRODUCTION

This work is related to Krasovskii's theory of guaranteeing positional control (see Krasovskii and Subbotin (1988), Subbotin and Chentsov (1981)). The theory focuses on assessment of the optimal guaranteed result – the minimax of a cost functional – for the control side that opposes the disturbance in the process of steering a dynamical control system. The properties of the optimal guaranteed result and an optimal strategy for the control side in the case where the non-observable dynamical disturbance subject to a functional constraint belonging to a given family of constraints.

Control problems with additional functional constraints imposed on the input dynamical disturbances have numerous interpretations and have been studied in various formalizations. The simplest functional constraint restricts the disturbances to the open-loop ones. In Krasovskii (1970), Krasovskii (1971), Krasovskii (1985) the maximin open loop constructions (including the stochastic ones) use open-loop disturbances to find the optimal guaranteed result and optimal closed-loop strategies in control problems with non-constrained disturbances. In Barabanova and Subbotin (1970), Barabanova and Subbotin (1971) properties of linear control systems in the cases of open-loop disturbances, disturbances generated by continuous feedbacks, and disturbances formed by upper semicontinuous set-valued closed-loop strategies were compared.

In Kryazhimskii (1991) assuming that the disturbances are restricted to an unknown  $L_2$ -compact set, it was shown that the optimal guaranteed result achieved in the class

of the full-memory closed-loop control strategies equals that achieved by the 'fully informed controller' allowed to control the system using quasi-strategies — nonanticipatory open-loop control responses to disturbance realizations Subbotin and Chentsov (1981); in this sense the full-memory closed-loop strategies are unimprovable. In Serkov (2013), considering the problem setting proposed in Kryazhimskii (1991) in the case of a continuous cost functional, new unimprovability conditions for full-memory control strategies were given and an optimal full-memory control strategy allowing numerical implementation was constructed.

For the case of a continuous cost functional in Serkov (2014) it was shown that, firstly, the guaranteed control problem with open-loop disturbances is equivalent to that with the disturbances restricted to  $L_2$ -compact sets, and, secondly, the optimal guaranteed results achieved by the controlling player in the class of full-memory control strategies under these two types of constraints on the disturbances are equal to that achieved in the class of quasi-strategies. In showing the results, the elements of theory of robust dynamical inversion of control systems (Kryazhimskii and Osipov (1983), Osipov and Kryazhimskii (1995)) were used. However, the control algorithm used in that study is not suitable for numerical implementation.

In this paper a new weakened condition on the control system (as compared with that in Kryazhimskii (1991) and Serkov (2013)), allowing numerical realization of an optimal full-memory control strategy, is provided.

Plan of the paper is as follows: in section 2 we give a formal statement of the problem and note the conditions of the works Kryazhinskii (1991) and Serkov (2013), in section 3, we present a solution of the problem, corresponding to the work Serkov (2014), and section 4 shall give the weakened conditions on the system and a modernization of the solutions suitable for further numerical implementation.

## 2. DEFINITIONS

Consider a control system

$$\begin{cases} \dot{x}(\tau) = f(\tau, x(\tau), u(\tau), v(\tau)), & \tau \in T = [t_0, \vartheta] \subset \mathbb{R}, \\ x(t_0) = z_0 \in G_0 \subset \mathbb{R}^n, \end{cases} \quad (1)$$

$$u(\tau) \in \mathcal{P} \subset \mathbb{R}^p, \quad v(\tau) \in \mathcal{Q} \subset \mathbb{R}^q, \quad \tau \in T.$$

Here  $\mathcal{P}$ ,  $\mathcal{Q}$ , and  $G_0$  are compact sets; and  $f(\cdot) : T \times \mathbb{R}^n \times \mathcal{P} \times \mathcal{Q} \mapsto \mathbb{R}^n$  is continuous, locally Lipschitz in the second argument and such that for some  $K \geq 0$  the inequality

$$\sup_{(\tau, u, v) \in T \times \mathcal{P} \times \mathcal{Q}} \|f(\tau, x, u, v)\| \leq K(1 + \|x\|)$$

holds for all  $x \in \mathbb{R}^n$  ( $\|\cdot\|$  denotes the norm in an Euclidian space). Controls  $u(\cdot) : T \mapsto \mathcal{P}$  and disturbances  $v(\cdot) : T \mapsto \mathcal{Q}$  are supposed to be Lebesgue measurable. Denote by  $\mathcal{U}$  the set of all controls and by  $\mathcal{V}$  the set of all disturbances.

For arbitrary  $(t_*, x_*) \in T \times \mathbb{R}^n$ ,  $u(\cdot) \in \mathcal{U}$ ,  $v(\cdot) \in \mathcal{V}$  we denote by  $x(\cdot, t_*, z_*, u(\cdot), v(\cdot))$  the (unique) Carathéodory solution of (1) (see (Warga, 1972, II.4)) defined on  $[t_*, \vartheta]$  and satisfying the initial condition  $x(t_*) = x_*$ . We fix a compact set  $G \subset T \times \mathbb{R}^n$  such that  $(t, x(t, t_0, z_0, u(\cdot), v(\cdot))) \in G$  for all  $t \in T$ ,  $z_0 \in G_0$ ,  $u(\cdot) \in \mathcal{U}$ ,  $v(\cdot) \in \mathcal{V}$ .

A set  $\Delta = (\tau_i)_{i \in 0..n_\Delta}$  where  $\tau_0 = t_0$ ,  $\tau_{i-1} < \tau_i$ ,  $\tau_{n_\Delta} = \vartheta$  will be called a *partition* (of interval  $T$ ). Denote by  $\Delta_T$  the set of all partitions. For a partition  $\Delta = (\tau_i)_{i \in 0..n_\Delta}$  and a  $t \in T$  set  $i_t = \max_{\tau_i \leq t} i$ ,  $D(\Delta) = \max_{i \in 1..n_\Delta} \tau_i - \tau_{i-1}$ .

Following Kryazhinskii (1991), define full-memory control strategies used by the controlling player. For every  $\tau_*, \tau^* \in T$  where  $\tau^* > \tau_*$  denote by  $\mathcal{U}_{[\tau_*, \tau^*)}$  the set of the restrictions of all controls to  $[\tau_*, \tau^*)$ . Given a partition  $\Delta = (\tau_i)_{i \in 0..n_\Delta}$ , any family  $\mathbf{U}^\Delta = (\mathbf{U}_i^\Delta(\cdot))_{i \in 0..(n_\Delta-1)}$ , where  $\mathbf{U}_i^\Delta(\cdot) : C([t_0, \tau_i], \mathbb{R}^n) \mapsto \mathcal{U}_{[\tau_i, \tau_{i+1})}$  ( $i \in 0..(n_\Delta-1)$ ), will be called a *full-memory feedback for partition*  $\Delta$ . Every family  $\mathbf{U} = (\mathbf{U}^\Delta)_{\Delta \in \Delta_T}$  where  $\mathbf{U}^\Delta$  is a full-memory feedback for  $\Delta$  will be called a *full-memory control strategy* (for the controlling player). We denote by  $\mathbf{S}$  the set of all full-memory control strategies.

Given a  $z_0 \in G_0$ , a partition  $\Delta = (\tau_i)_{i \in 0..n_\Delta}$ , a full-memory feedback  $\mathbf{U}^\Delta = (\mathbf{U}_i^\Delta(\cdot))_{i \in 0..(n_\Delta-1)}$  for  $\Delta$  and a disturbance  $v(\cdot) \in \mathcal{V}$ , the function  $x(\cdot) = x(\cdot, t_0, z_0, u(\cdot), v(\cdot))$  where  $u(\cdot) \in \mathcal{U}$  is such that  $u(t) = \mathbf{U}_{i_t}^\Delta(x(\cdot)|_{[t_0, \tau_{i_t}]})(t)$  for all  $t \in T$  will be called the (system's) *motion* originating at  $z_0$  and corresponding to  $\Delta$ ,  $\mathbf{U}^\Delta$  and  $v(\cdot)$ ; we denote  $x(\cdot)$  and  $u(\cdot)$  by  $x(\cdot, z_0, \mathbf{U}^\Delta, v(\cdot))$  and  $u(\cdot, z_0, \mathbf{U}^\Delta, v(\cdot))$ , respectively.

For every  $z_0 \in G_0$ , every full-memory control strategy  $\mathbf{U} = (\mathbf{U}^\Delta)_{\Delta \in \Delta_T} \in \mathbf{S}$  and every nonempty set of disturbances,  $\mathbf{V} \subseteq \mathcal{V}$ , we define the *bundle of motions* originating from  $z_0$  and corresponding to  $\mathbf{U}$  and  $\mathbf{V}$  to be the set  $X(z_0, \mathbf{U}, \mathbf{V})$  of all  $x(\cdot) \in C(T; \mathbb{R}^n)$  with the following property: there is

a sequence  $\{(z_{0k}, v_k(\cdot), \Delta_k, \mathbf{U}^{\Delta_k})\}_{k=1}^\infty$  in  $G_0 \times \mathbf{V} \times \Delta_T \times \mathbf{U}$  such that  $\lim_{k \rightarrow \infty} z_{0k} = z_0$ ,  $\lim_{k \rightarrow \infty} D(\Delta_k) = 0$  and  $x(\cdot, z_{0k}, \mathbf{U}^{\Delta_k}, v_k(\cdot)) \rightarrow x(\cdot)$  in  $C(T; \mathbb{R}^n)$ .

In the above definition,  $\mathbf{V}$  is a functional constraint on the disturbance. Generally, we assume that the controlling side does not know  $\mathbf{V}$  but knows a class of functional constraints  $\mathbf{V}$  belongs to. The latter class gives a general information on the disturbing constraints but does not provide information on the exact ones.

In the above cases, for every  $z_0 \in G_0$  and every full-memory control strategy  $\mathbf{U} \in \mathbf{S}$ , we define the bundles of the system's *motions* originating at  $z_0$  under  $\mathbf{U}$  subject to *arbitrary disturbances*,  $L_p$ -*compactly constrained disturbances* (for some fixed  $p > 1$ ), and *open-loop disturbances* as, respectively,

$$\mathcal{X}(z_0, \mathbf{U}) = X(z_0, \mathbf{U}, \mathbf{V}),$$

$$\mathcal{X}_c(z_0, \mathbf{U}) = \bigcup_{\mathbf{V} \in \mathbf{comp}_{L_p(T, \mathbb{R}^q)}(\mathcal{V})} X(z_0, \mathbf{U}, \mathbf{V}),$$

$$\mathcal{X}_p(z_0, \mathbf{U}) = \bigcup_{v(\cdot) \in \mathcal{V}} X(z_0, \mathbf{U}, \{v(\cdot)\});$$

here  $\mathbf{comp}_{L_p(T, \mathbb{R}^q)}(\mathcal{V})$  denotes the family of all subsets of  $\mathcal{V}$  compact in  $L_p(T; \mathbb{R}^q)$ .

*Remark 1.* The definition of  $\mathcal{X}(z_0, \mathbf{U})$  is a straightforward generalization of the definition of the set of constructive motions generated by a closed-loop control strategy (see Krasovskii and Subbotin (1988)). The definition of  $\mathcal{X}_c(z_0, \mathbf{U})$  follows Kryazhinskii (1991).

According to the definitions,  $\mathcal{X}_p(z_0, \mathbf{U}) \subseteq \mathcal{X}_c(z_0, \mathbf{U}) \subseteq \mathcal{X}(z_0, \mathbf{U})$  holds for all  $z_0 \in G_0$  and  $\mathbf{U} \in \mathbf{S}$ . In Serkov (2009) it was shown that generally  $\mathcal{X}_p(z_0, \mathbf{U}) \neq \mathcal{X}(z_0, \mathbf{U})$ . A similar reasoning can lead to a statement that generally  $\mathcal{X}_c(z_0, \mathbf{U}) \neq \mathcal{X}(z_0, \mathbf{U})$ .

Let the controlling player evaluate the quality of the system's motions by a continuous *cost functional*  $\gamma(\cdot) : C(T; \mathbb{R}^n) \mapsto \mathbb{R}$  acting as a benefit functional for the disturbing player. The controlling player seeks then to chose a full-memory control strategy that guarantees the minimum value for the supremum of  $\gamma(x(\cdot))$  over the system's motions  $x(\cdot)$  corresponding to the chosen control strategy and all disturbances that are allowed to be chosen by the disturbing player within the given constraints.

Following Krasovskii and Subbotin (1988), and Subbotin and Chentsov (1981), we call

$$\Gamma(z_0, \mathbf{U}) = \sup_{x(\cdot) \in \mathcal{X}(z_0, \mathbf{U})} \gamma(x(\cdot))$$

the *guaranteed result* at  $z_0 \in G_0$  for a full-memory control strategy  $\mathbf{U}$  against *arbitrary disturbances*; and we call

$$\Gamma(z_0) = \inf_{\mathbf{U} \in \mathbf{S}} \Gamma(z_0, \mathbf{U})$$

the *optimal guaranteed result* at  $z_0 \in G_0$  in the class of the full-memory control strategies  $\mathbf{S}$ , against *arbitrary disturbances*. Similarly, we call

$$\Gamma_c(z_0, \mathbf{U}) = \sup_{x(\cdot) \in \mathcal{X}_c(z_0, \mathbf{U})} \gamma(x(\cdot))$$

the *guaranteed result* at  $z_0 \in G_0$  for a full-memory control strategy  $\mathbf{U}$  against  $L_p$ -*compactly constrained disturbances* and we call

$$\Gamma_c(z_0) = \inf_{\mathbb{U} \in \mathbf{S}} \Gamma_c(z_0, \mathbb{U})$$

the *optimal guaranteed result* at  $z_0 \in G_0$  in  $\mathbf{S}$  against  $L_P$ -compactly constrained disturbances. Finally, we call

$$\Gamma_P(z_0, \mathbb{U}) = \sup_{x(\cdot) \in \mathcal{X}_r(z_0, \mathbb{U})} \gamma(x(\cdot)).$$

the *guaranteed result* at  $z_0 \in G_0$  for a full-memory control strategy  $\mathbb{U}$  against *open-loop disturbances*; and we call

$$\Gamma_P(z_0) = \inf_{\mathbb{U} \in \mathbf{S}} \Gamma_P(z_0, \mathbb{U}).$$

the *optimal guaranteed result* at  $z_0 \in G_0$  in  $\mathbf{S}$  against *open-loop disturbances*.

Along with the full-memory control strategies, we introduce, after Subbotin and Chentsov (1981), control quasi-strategies — nonanticipatory transformations of disturbances into controls. The controlling player uses quasi-strategies if he/she is fully informed about the current histories and current values of the disturbance. A *control quasi-strategy* is a mapping  $\alpha(\cdot) : \mathcal{V} \mapsto \mathcal{U}$  satisfying the following condition:  $\alpha(v(\cdot))|_{[t_0, \tau]} = \alpha(v'(\cdot))|_{[t_0, \tau]}$  for any  $\tau \in T$ ,  $v(\cdot), v'(\cdot) \in \mathcal{V}$  such that  $v(\cdot)|_{[t_0, \tau]} = v'(\cdot)|_{[t_0, \tau]}$ . We denote by  $\mathbf{Q}$  the set of all control quasi-strategies. For every  $z_0 \in G_0$  and every control quasi-strategy  $\alpha(\cdot)$ , we call

$$\mathcal{X}(z_0, \alpha(\cdot)) = \{x(\cdot, t_0, z_0, \alpha(v(\cdot)), v(\cdot)) \mid v(\cdot) \in \mathcal{V}\}$$

the bundle of *motions* originating at  $z_0$  under  $\alpha(\cdot)$ . For every  $z_0 \in G_0$  the value

$$\Gamma_Q(z_0, \alpha(\cdot)) = \sup_{x(\cdot) \in \mathcal{X}(z_0, \alpha(\cdot))} \gamma(x(\cdot))$$

is called the *guaranteed result* at  $z_0$  for a control-quasi-strategy  $\alpha(\cdot)$  against *arbitrary disturbances*, and

$$\Gamma_Q(z_0) = \inf_{\alpha(\cdot) \in \mathbf{Q}} \Gamma_Q(z_0, \alpha(\cdot))$$

is called the *optimal guaranteed result* at  $z_0$  in the class of the control quasi-strategies,  $\mathbf{Q}$ , against *arbitrary disturbances*.

**Theorem 2.** For every  $z_0 \in G_0$

$$\Gamma_Q(z_0) \leq \Gamma_P(z_0) \leq \Gamma_c(z_0) \leq \Gamma(z_0). \quad (2)$$

**Remark 3.** From the results Krasovskii and Subbotin (1988), and Subbotin and Chentsov (1981) follows, that for every  $z_0 \in G_0$  all the inequalities in (2) turn into equalities if

$$\min_{u \in \mathcal{P}} \max_{v \in \mathcal{Q}} \langle s, f(\tau, x, u, v) \rangle = \max_{v \in \mathcal{Q}} \min_{u \in \mathcal{P}} \langle s, f(\tau, x, u, v) \rangle \quad (3)$$

for all  $(\tau, x) \in G$ ,  $s \in \mathbb{R}^n$ . In that case neither the  $L_P$ -compact, nor open-loop constraints on the disturbances change the optimal guaranteed result.

In this paper we do not assume (3) to be satisfied for all  $(\tau, x) \in G$ ,  $s \in \mathbb{R}^n$ . In such circumstances, some inequalities given in (2) can be strict. Examples of the situations where the first and last elements in the chain (2) differ are well known (see (Subbotin and Chentsov, 1981, Chapter VI, §1)). For the case where the cost functional  $\gamma$  is uniformly  $(L^1, \delta)$ -continuous on the set of all motions of system (1) but is not continuous on  $C(T, \mathbb{R}^n)$ , an example of the situation where the last inequality in (2) is strict, was constructed in Kryazhinskii (1991) (where one can also find a definition of the uniform  $(L^1, \delta)$ -continuity). For  $\gamma$  continuous on  $C(T, \mathbb{R}^n)$  a similar example was given in Serkov (2010).

Among the optimal guaranteed results (at a  $z_0 \in G_0$ ) given in (2) the smallest one is the optimal guaranteed result in the class of the control quasi-strategies. We address a question whether the optimal guaranteed result (at  $z_0$ ) in the class of the full-memory control strategies against either open-loop disturbances, or  $L_P$ -compactly constrained disturbances coincides with that in the class of quasi-strategies. If the answer is positive, the class of the full-memory control strategies,  $\mathbf{S}$ , is *non-improvable* against a corresponding type of functional constraints on the disturbances. In that situation, the use of any information on the past and current values of the actual disturbance does not allow the controlling player to improve the value of the optimal guaranteed result at any  $z_0 \in G_0$ , provided the disturbing player's choices are subject to the corresponding type of functional constraints.

In Kryazhinskii (1991) it was shown that in the case of a uniformly  $(L^1, \delta)$ -continuous cost functional the one-to-one correspondence in the mapping  $v \mapsto f(t, x, u, v)$  for all  $(t, x, u) \in T \times \mathbb{R}^n \times \mathcal{P}$  is sufficient for the non-improvability of  $\mathbf{S}$  against  $L_2$ -compactly constrained disturbances.

In Serkov (2013) for the case of a cost functional continuous in  $C(T, \mathbb{R}^n)$  the following sufficient condition for the non-improvability of  $\mathbf{S}$  against the  $L_2$ -compactly constrained disturbances was given: for all  $(t, x, u) \in G \times \mathcal{P}$  we denote  $\mathcal{Q}_{txu}$  the quotient set of the set  $\mathcal{Q}$ , generated by the equivalence relation  $\sim_{txu} : (v_1 \sim_{txu} v_2) \Leftrightarrow (f(t, x, u, v_1) = f(t, x, u, v_2))$ . The condition consisted in independency of  $\mathcal{Q}_{txu}$  on  $u \in \mathcal{P}$ :

$$\mathcal{Q}_{txu} = \mathcal{Q}_{txu'} \quad \text{for all } u, u' \in \mathcal{P}, (t, x) \in G. \quad (4)$$

In Serkov (2014) for the case of a cost functional continuous in  $C(T, \mathbb{R}^n)$  demonstrated, that the first and second relations in (2) turns into an equality, without any additional condition to be assumed. So, it is shown that at every  $z_0 \in G_0$  the class of the full-memory control strategies,  $\mathbf{S}$ , is non-improvable against both  $L_P$ -compactly constrained and open-loop disturbances.

### 3. NON-IMPROVABILITY OF FULL-MEMORY CONTROL STRATEGIES

In this section we construct a family  $(\mathbb{U}_\varepsilon)_{\varepsilon > 0}$  of full-memory control strategies,  $\mathbb{U}_\varepsilon = (\mathbf{U}_\varepsilon^\Delta)_{\Delta \in \Delta_T}$  ( $\varepsilon > 0$ ), such that for a given  $z_0 \in G_0$

$$\limsup_{\varepsilon \rightarrow 0} \Gamma_c(z_0, \mathbb{U}_\varepsilon) \leq \Gamma_Q(z_0).$$

Then, in view of (2), we get

$$\Gamma_c(z_0) = \Gamma_P(z_0) = \Gamma_Q(z_0),$$

which implies that the full-memory control strategies are non-improvable at  $z_0$  against both  $L_2$ -compactly constrained and open-loop disturbances.

The process of operation of the full-memory feedback  $\mathbf{U}_\varepsilon^\Delta = (\mathbf{U}_{\varepsilon i}^\Delta(\cdot))_{i \in 0..(n_\Delta - 1)}$  for a partition  $\Delta = (\tau_i)_{i \in 0..n_\Delta}$  includes on-line simulation of a motion  $y(\cdot)$  of an auxiliary copy of system (1), which we call the *y-model*, on every interval  $[\tau_i, \tau_{i+1})$ . In the simulation process, the control side implements the robust dynamical inversion approach (Kryazhinskii and Osipov (1983); Osipov and Kryazhinskii (1995)). He/she identifies a 'surrogate' disturbance  $\bar{v}_i$  that mimics the affect of the actual disturbance on the

system, and lets the 'surrogate' disturbance operate in the  $y$ -model. To identify the 'surrogate' disturbance  $\bar{v}_i$ , in a small final part of the time interval  $[\tau_{i-1}, \tau_i]$  the controlling player implements a series of test control actions  $u_1^\varepsilon, \dots, u_{n_\varepsilon}^\varepsilon$  and observes the system's reactions driven by the actual disturbance. In the major initial part of  $[\tau_i, \tau_{i+1}]$  the controlling player implements the useful control action  $u_i$  constructed as the optimal response to the 'surrogate' disturbance for the  $y$ -model, whereas the latter is driven by the useful control action  $u_{i-1}$  and 'surrogate' disturbance  $\bar{v}_{i-1}$  formed previously. The optimal response  $u_i$  is found using Krasovskii's extremal shift principle (Krasovskii and Subbotin (1988));  $u_i$  shifts the  $y$ -model to a target set at the maximum speed. The target set is formed in advance and comprises the histories (up to time  $\tau_i$ ) of the uniform limits of the system's motions corresponding to 'approximately optimal' control quasi-strategies. The above control process ensures that the current histories of both the system's and  $y$ -model's motions never abandon small neighborhoods of the current target sets, implying that at the final time,  $\vartheta$ , the value of the cost functional does not exceed  $\Gamma_Q(z_0) + \varphi(\varepsilon)$  for some  $\varphi(\cdot)$  satisfying  $\varphi(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$ .

Now we turn to formal definitions. In the construction of the target sets we use the system's motions corresponding to 'approximately optimal' control quasi-strategies. We set

$$\mathcal{W}(z) = \bigcap_{\delta > 0} \text{cl} \bigcup_{\Gamma_Q(z, \alpha(\cdot)) \leq \Gamma_Q(z) + \delta} \mathcal{X}(z, \alpha(\cdot));$$

here  $\text{cl} X$  denotes the closure of a  $X \subset C(T; \mathbb{R}^n)$  in  $C(T; \mathbb{R}^n)$ . For every  $\tau \in T$  the set of the restrictions of all the elements of  $\mathcal{W}(z)$  to  $[t_0, \tau]$ , denoted by  $\mathcal{W}(z)|_{[t_0, \tau]}$ , will be regarded as the *target set* at time  $\tau$ . For every  $\tau \in T$  and every  $y(\cdot) \in C([t_0, \tau], \mathbb{R}^n)$  we fix a projection  $w(\cdot | \tau, y(\cdot))$  of  $y(\cdot)$  onto the target set  $\mathcal{W}(z)|_{[t_0, \tau]}$ ; thus,

$$w(\cdot | \tau, y(\cdot)) \in \underset{w(\cdot) \in \mathcal{W}(y(t_0))|_{[t_0, \tau]}}{\text{argmin}} \|w(\cdot) - y(\cdot)\|_{C([t_0, \tau], \mathbb{R}^n)}. \quad (5)$$

Fix an  $\varepsilon \in (0, 1)$ . Fix an  $\varepsilon$ -net  $(u_j^\varepsilon)_{j \in 1..n_\varepsilon}$  in  $\mathcal{P}$ ; thus,  $\sup_{u \in \mathcal{P}} \min_{j \in 1..n_\varepsilon} \|u - u_j^\varepsilon\| \leq \varepsilon$ . In the subsequent constructions the elements of  $(u_j^\varepsilon)_{j \in 1..n_\varepsilon}$  play the role of test control actions mentioned above.

Let  $\Delta = (\tau_i)_{i \in 0..n_\Delta}$  be a partition of  $T$ . For simplicity we give the definitions for the case of partition with constant step. Denote

$$\tau'_i = \tau_i - \varepsilon \Delta(\Delta), \quad i \in 1..(n_\Delta - 1), \quad (6)$$

$$\tau'_{ij} = \tau'_i + \frac{j(\tau_i - \tau'_i)}{n_\varepsilon}, \quad j \in 0..n_\varepsilon, \quad i \in 1..(n_\Delta - 1). \quad (7)$$

For every  $x(\cdot) \in C(T; \mathbb{R}^n)$ ,  $j \in 1..n_\varepsilon$ ,  $i \in 1..(n_\Delta - 1)$  let

$$d_{ij}(x(\cdot)) = \frac{x(\tau'_{ij}) - x(\tau'_{i(j-1)})}{\tau'_{ij} - \tau'_{i(j-1)}}. \quad (8)$$

Define a full-memory feedback  $\mathbf{U}_\varepsilon^\Delta = (\mathbf{U}_{\varepsilon i}^\Delta(\cdot))_{i \in 0..(n_\Delta - 1)}$  for  $\Delta$  inductively. Fix some  $u_* \in \mathcal{P}$ ,  $v_* \in \mathcal{Q}$ . For every  $x_0(\cdot) \in C([t_0, \tau_0], \mathbb{R}^n)$  (recall that  $\tau_0 = t_0$ ) we set

$$\bar{v}_0 = v_*, \quad u_0 = u_*, \quad y_0(\tau_0) = z_0, \quad (9)$$

$$\mathbf{U}_{\varepsilon 0}^\Delta(x_0(\cdot))(t) = \begin{cases} u_0, & t \in [\tau_0, \tau'_1], \\ u_j^\varepsilon, & t \in [\tau'_{1(j-1)}, \tau'_{1j}], \quad j \in 1..n_\varepsilon. \end{cases} \quad (10)$$

If for some  $i \in 1..(n_\Delta - 1)$  elements

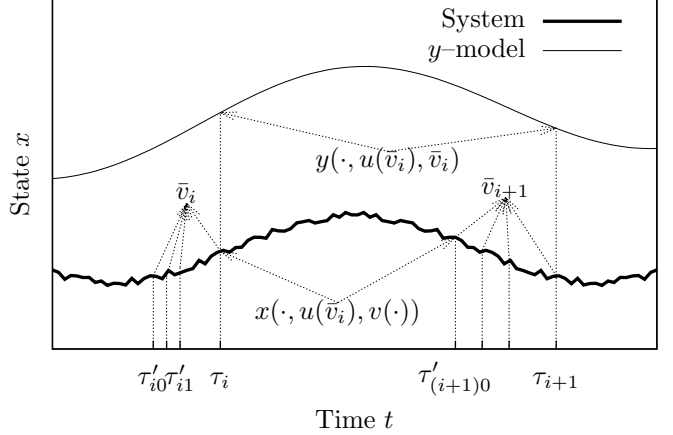


Fig. 1. The scheme of the strategy  $\mathbb{U}_\varepsilon$ .

$\bar{v}_{i-1} = \bar{v}_{i-1}(x_{i-1}(\cdot)) \in \mathcal{Q}$ ,  $\mathbf{U}_{\varepsilon(i-1)}^\Delta(x_{i-1}(\cdot)) \in \mathcal{U}|_{[\tau_{i-1}, \tau_i]}$ , and

$$y_{i-1}(\cdot) = y_{i-1}(\cdot, x_{i-1}(\cdot)) \in C([t_0, \tau_{i-1}], \mathbb{R}^n)$$

(a motion of the  $y$ -model on  $[t_0, \tau_{i-1}]$ ) are defined for all  $x_{i-1}(\cdot) \in C([t_0, \tau_{i-1}], \mathbb{R}^n)$ , then for every  $x_i(\cdot) \in C([t_0, \tau_i], \mathbb{R}^n)$  we define  $y_i(\cdot) = y_i(\cdot, x_i(\cdot)) \in C([t_0, \tau_i], \mathbb{R}^n)$  as the extension of  $y_{i-1}(\cdot)$  to  $[t_0, \tau_i]$  such that

$$y_i(\tau) = y_{i-1}(\tau_{i-1}, x_i(\cdot)|_{[t_0, \tau_{i-1}]}) + \int_{\tau_{i-1}}^\tau f(t, y_i(t), \mathbf{U}_{\varepsilon(i-1)}^\Delta(x_i(\cdot)|_{[t_0, \tau_{i-1}]}) (\tau_{i-1}), \bar{v}_{i-1}(x_i(\cdot)|_{[t_0, \tau_{i-1}]})) dt, \quad \tau \in [\tau_{i-1}, \tau_i], \quad (11)$$

and set

$$\bar{v}_i \in \underset{v \in \mathcal{Q}}{\text{argmin}} \max_{j \in 1..n_\varepsilon} \|d_{ij}(x_i(\cdot)) - f(\tau_i, x_i(\tau_i), u_j^\varepsilon, v)\|, \quad (12)$$

$$u_i \in \underset{u \in \mathcal{P}}{\text{argmin}} \langle y_i(\tau_i) - w(\tau_i | \tau_i, y_i(\cdot)), f(\tau_i, y_i(\tau_i), u, \bar{v}_i) \rangle, \quad (13)$$

$$\mathbf{U}_{\varepsilon i}^\Delta(x_i(\cdot))(t) = \begin{cases} u_i, & t \in [\tau_i, \tau'_{i+1}), \\ u_j^\varepsilon, & t \in [\tau'_{(i+1)(j-1)}, \tau'_{(i+1)j}), j \in 1..n_\varepsilon. \end{cases} \quad (14)$$

The full-memory feedback  $\mathbf{U}_\varepsilon^\Delta$  is defined for the partition  $\Delta \in \Delta_T$ . Thus, the full-memory strategy  $\mathbb{U}_\varepsilon = (\mathbf{U}_\varepsilon^\Delta)_{\Delta \in \Delta_T}$  is defined.

Illustration of the proposed control scheme is shown in Figure 1.

As it was mentioned, the following theorem holds true.

**Theorem 4.** For all  $z_0 \in G_0$  the relations

$$\limsup_{\varepsilon \rightarrow 0} \Gamma_c(z_0, \mathbb{U}_\varepsilon) \leq \Gamma_Q(z_0), \quad (15)$$

$$\Gamma_P(z_0) = \Gamma_c(z_0) = \Gamma_Q(z_0) \quad (16)$$

are fulfilled.

#### 4. THE CONSTRUCTIVE MODIFICATIONS OF THE OPTIMAL STRATEGY

In the construction of the strategy  $\mathbb{U}_\varepsilon$  there are at least two places, that may constitute essential difficulties when trying to implement numerically this control procedure.

The first is related to the calculation of projections of movements of  $y$ -model onto the 'target' set (see (5)). Conceptually the problem reduces to calculation of the gradient of 'lower' (maximin) game value in the current state of the control system. Despite the difficulty of this problem, it has long been known, extensively investigated and has viable solutions in many important cases.

The second difficulty is unlimited and fairly rapid growth of sets  $(u_j^\varepsilon)_{j \in 1..n_\varepsilon}$  with decreasing of parameter  $\varepsilon$ . This leads to significant increase in the dimension of minimization problem (12). Below a sufficiently broad class of systems is provided, wherein the difficulty can be overcome.

For any  $(\tau, x, u, v) \in G \times \mathcal{P} \times \mathcal{Q}$  denote the  $q_{txu}(v)$  the (unique) element of  $\mathcal{Q}_{txu}$ , containing  $v$ .

*Assumption 5.* There is a finite subset  $\{\bar{u}_j \in \mathcal{P} \mid j \in 1..l\}$  such that for all  $(\tau, x, u, v) \in G \times \mathcal{P} \times \mathcal{Q}$  holds

$$\bigcap_{j \in 1..l} q_{tx\bar{u}_j}(v) \subseteq q_{txu}(v). \quad (17)$$

*Remark 6.* The assumption implies that for every  $v \in \mathcal{Q}$  response of the system to the control  $u \in \mathcal{P}$  can be calculated knowing the response of the system at the same  $v$  on the final set of test control actions  $\{\bar{u}_j \in \mathcal{P} \mid j \in 1..l\}$ . And so, to select an approximating value  $\bar{v}$  (see (12)) this final set is enough. It is easy to see that this condition generalizes the condition (4).

We define a family of strategies  $(\bar{\mathbf{U}}_\varepsilon)_\varepsilon$  ( $\bar{\mathbf{U}}_\varepsilon \in \mathbf{S}$ ,  $\varepsilon > 0$ ),  $\bar{\mathbf{U}}_\varepsilon = (\bar{\mathbf{U}}_\varepsilon^\Delta)_{\Delta \in \Delta_T}$ , where for every  $\Delta \in \Delta_T$  the feedback with full memory  $\bar{\mathbf{U}}_\varepsilon^\Delta$  is defined by relations (6)–(14), wherein  $n_\varepsilon = l$  and  $u_j^\varepsilon = \bar{u}_j$ ,  $j \in 1..n_\varepsilon$ .

*Theorem 7.* Let the controlled system (1) satisfies Assumption 5. Then, for all  $z_0 \in G_0$  the following equalities hold

$$\limsup_{\varepsilon \rightarrow 0} \Gamma_c(z_0, \bar{\mathbf{U}}_\varepsilon) = \Gamma_c(z_0). \quad (18)$$

*Remark 8.* From the construction it is clear that in case of the conditions from the theorem 7, in the problem of inverse dynamics (12) the data amount is fixed.

The proof of Theorem 7 in its basic steps follows the proof of Theorem 4.

Another control strategy  $\mathbb{U}_*$  (see (Serkov, 2013, Theorem 2)) uses the value of control at the previous step as the only 'test control action'. In terms of this work that means  $\tau_i' = \tau_{i-1}$ ,  $n_\varepsilon = 1$ ,  $u_1^\varepsilon = u_{i-1}$ . Thanks to condition (4) it was enough to identify a surrogate disturbance. This construction can be generalized by using instead of condition (4) the following assumption:

*Assumption 9.* There exists a closed subset  $\bar{\mathcal{P}} \subseteq \mathcal{P}$  such that for all  $(\tau, x, v, s) \in G \times \mathcal{Q} \times \mathbb{R}^n$ ,  $u, u' \in \bar{\mathcal{P}}$  the following relations hold

$$\begin{aligned} \operatorname{argmin}_{u \in \bar{\mathcal{P}}} \langle s, f(\tau, x, u, v) \rangle \cap \bar{\mathcal{P}} &\neq \emptyset, \\ \mathcal{Q}_{txu} &= \mathcal{Q}_{txu'}. \end{aligned} \quad (19)$$

Let define this modification of the strategy  $\mathbb{U}_* = (\mathbf{U}_*^\Delta)_{\Delta \in \Delta_T}$  formally: define a full-memory feedback  $\mathbf{U}_*^\Delta = (\mathbf{U}_{*i}^\Delta(\cdot))_{i \in 0..(n_\Delta-1)}$  for  $\Delta$  inductively. For every  $x_0(\cdot) \in C([t_0, \tau_0], \mathbb{R}^n)$  we set

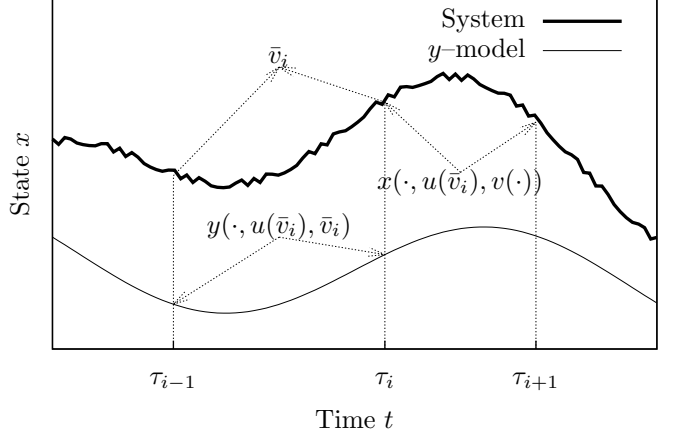


Fig. 2. The scheme of the strategy  $\mathbb{U}_*$ .

$$\bar{v}_0 = v_*, \quad u_0 = u_*, \quad y_0(\tau_0) = z_0, \quad (20)$$

$$\mathbf{U}_{*0}^\Delta(x_0(\cdot))(t) = u_0, \quad t \in [\tau_0, \tau_1]. \quad (21)$$

If for some  $i \in 1..(n_\Delta - 1)$  elements

$$\bar{v}_{i-1} = \bar{v}_{i-1}(x_{i-1}(\cdot)) \in \mathcal{Q}, \quad \mathbf{U}_{*(i-1)}^\Delta(x_{i-1}(\cdot)) \in \mathcal{U}_{[\tau_{i-1}, \tau_i]},$$

and

$$y_{i-1}(\cdot) = y_{i-1}(\cdot, x_{i-1}(\cdot)) \in C([t_0, \tau_{i-1}], \mathbb{R}^n)$$

are defined for all  $x_{i-1}(\cdot) \in C([t_0, \tau_{i-1}], \mathbb{R}^n)$ , then for every  $x_i(\cdot) \in C([t_0, \tau_i], \mathbb{R}^n)$  we define  $y_i(\cdot) = y_i(\cdot, x_i(\cdot)) \in C([t_0, \tau_i], \mathbb{R}^n)$  as the extension of  $y_{i-1}(\cdot)$  to  $[t_0, \tau_i]$  such that

$$\begin{aligned} y_i(\tau) &= y_{i-1}(\tau_{i-1}, x_i(\cdot)|_{[t_0, \tau_{i-1}]}) \\ &+ \int_{\tau_{i-1}}^\tau f(t, y_i(t), \mathbf{U}_{*i-1}^\Delta(x_i(\cdot)|_{[t_0, \tau_{i-1}]}) (\tau_{i-1}), \\ &\bar{v}_{i-1}(x_i(\cdot)|_{[t_0, \tau_{i-1}]}) dt, \quad \tau \in [\tau_{i-1}, \tau_i], \end{aligned} \quad (22)$$

and set

$$\begin{aligned} \bar{v}_i \in \operatorname{argmin}_{v \in \mathcal{Q}} &\left\| \frac{x_i(\tau_i) - x_i(\tau_{i-1})}{\tau_i - \tau_{i-1}} \right. \\ &\left. - f(\tau_i, x_i(\tau_i), \mathbf{U}_{*i-1}^\Delta(x_i(\cdot))(\tau_{i-1}), v) \right\|, \end{aligned} \quad (23)$$

$$u_i \in \operatorname{argmin}_{u \in \mathcal{P}} \langle y_i(\tau_i) - w(\tau_i \mid \tau_i, y_i(\cdot)), f(\tau_i, y_i(\tau_i), u, \bar{v}_i) \rangle, \quad (24)$$

$$\mathbf{U}_{*i}^\Delta(x_i(\cdot))(t) = u_i, \quad t \in [\tau_i, \tau_{i+1}]. \quad (25)$$

The full-memory feedback  $\mathbf{U}_*^\Delta$  is defined for the partition  $\Delta \in \Delta_T$ . So, the full-memory strategy  $\mathbb{U}_* = (\mathbf{U}_*^\Delta)_{\Delta \in \Delta_T}$  is defined. The scheme of the control strategy is shown in Figure 2.

*Theorem 10.* Let the controlled system (1) satisfies Assumption 9. Then, the equalities

$$\Gamma_c(z_0, \mathbb{U}_*) = \Gamma_c(z_0) \quad (26)$$

are fulfilled for all  $z_0 \in G_0$ .

## 5. EXAMPLE

Let system (1) have the form

$$\begin{cases} \dot{x}_1(\tau) = u_1(\tau)v_1(\tau), & \tau \in T = [0, 1], \\ \dot{x}_2(\tau) = \max\{0, x_1(\tau)\}u_2(\tau)v_2(\tau), \\ (x_1(0), x_2(0)) = (0, 0), & G_0 = \{(0, 0)\}, \end{cases} \quad (27)$$

$u_1(\tau), u_2(\tau) \in [-1, 1], \quad v_1(\tau), v_2(\tau) \in \{-1, 1\}$ ,  
and the cost functional be given by  $\gamma(x(\cdot)) = x_2(1)$  ( $x(\cdot) = (x_1(\cdot), x_2(\cdot)) \in C(T, \mathbb{R}^2)$ ). With an appropriate choice of  $G$ , system (27) satisfies all the assumptions imposed earlier on system (1); therefore, Theorem 4 holds, implying the full-memory control strategies are non-improvable against both the  $L_2$ -compactly constrained and open-loop disturbances. On the other hand, system (1) does not satisfy the conditions sufficient for the non-improvability of the full-memory control strategies against the  $L_2$ -compactly constrained disturbances, which are given in Kryazhinskii (1991) (Theorem 9.1) and in Serkov (2013) (Theorem 2). By use of relations (16) one can find that  $\Gamma_c((0, 0)) = \Gamma_p((0, 0)) = \Gamma_q((0, 0)) = -0.5$ .

It is clear that the set  $\bar{\mathcal{P}} = \{-1, 1\} \subset \mathcal{P}$  satisfies to both assumptions 5, 9. So, we can use the construction of optimal strategy  $\mathbb{U}_* = (\mathbf{U}_*^\Delta)_{\Delta \in \Delta_T}$  given in (20)–(25): by using the monotonicity of the quality index  $\gamma$ , we get the feedback with full memory  $\mathbf{U}_*^\Delta = (\mathbf{U}_{*i}^\Delta)_{i \in 0..n_\Delta-1}$  for the partition  $\Delta = (\tau_i)_{i \in 0..n_\Delta}$ , the motion  $(x(\cdot) \in C([t_0, \tau_i], \mathbb{R}^2))$  and  $i \in 1..(n_\Delta - 1)$ :

$$\mathbf{U}_{*i}^\Delta(x(\cdot)) \in \begin{pmatrix} \operatorname{argmax}_{u_1 \in \bar{\mathcal{P}}} \left\{ u_1 \cdot \frac{x_1(\tau_i) - x_1(\tau_{i-1})}{u_1(\tau_{i-1})} \right\} \\ \operatorname{argmin}_{u_2 \in \bar{\mathcal{P}}} \left\{ u_2 \cdot \frac{x_2(\tau_i) - x_2(\tau_{i-1})}{u_2(\tau_{i-1})} \right\} \end{pmatrix},$$

$$(u_1(\tau_{i-1}), u_2(\tau_{i-1})) = \mathbf{U}_{*(i-1)}^\Delta(x(\cdot)|_{[t_0, \tau_{i-1}]}).$$

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